Low-Complexity Non-Binary Forward Error Correction for Lattice-Based 4D Constellations

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Abstract: Low-complexity non-binary LDPC decoding is studied for a 512-ary lattice-based 4D Welti constellation. In an 800ZR scenario, more than 1 dB SNR gain is obtained over DP-16QAM and binary FEC at fixed symbol rate.

1. Introduction

Implementation agreements (IAs) such as the IA 400ZR [1] and its upcoming successor 800ZR specify coherent low-power links between data centers with ranges of around 100 km. In such scenarios, a high power efficiency of the (coded) modulation scheme is desirable to enable low transmit powers and to avoid non-linear effects. Moreover, high power consumption due to large signal-processing effort should be avoided at the pluggable modules.

Probabilistic shaping has become a popular technique to increase the power efficiency [2]. Nevertheless, at fixed spectral efficiency, it is accompanied by an increase in either the symbol rate or the signal constellation's cardinality. Constellations optimized w.r.t. achievable mutual information (MI) can be used alternatively [2]. However, they usually have a non-uniform spacing; a drawback, e.g., if low-complexity D/A and A/D converters are applied.

In dual-polarization (DP) transmission, 4D constellations based on the D_4 lattice [3–5] are convenient. Keeping the symbol rate constant, they enable a *packing gain* due to an increased number of signal points at fixed minimum distance in comparison to DP quadrature-amplitude modulation (DP-QAM) [5], and additionally a *shaping gain* if the constellation is bounded by a 4D hypersphere instead of a square [3, 6]. However, a Gray labeling cannot be given, i.e., *binary* forward error correction (FEC) using bit-interleaved coded modulation (BICM) results in a large non-Gray penalty for the BICM-MI. This penalty can be avoided by multilevel coding [5] (MLC), i.e., by the decomposition of the constellation into its *bit levels* which are protected by individual component codes. To that end, interim decoding results have to be buffered for multi-stage decoding, an issue for high data rates.

Recently, a related *concatenated non-binary* FEC scheme has been proposed [6, 7]: Instead of individual *binary* coding, 4 bit levels are *jointly* protected by an inner *non-binary* low-density parity-check (LDPC) code. All other levels are uncoded as they are highly reliable due to a 4D set partitioning. Residual bit errors are handled by outer hard-decision (HD) decoding. For inner soft-decision (SD) decoding, only a single, 16-ary LDPC code is required. However, 16-ary belief-propagation (BP) decoding is still too complex for low-power scenarios [8]. Hence, in this work, concatenated FEC with *low-complexity non-binary decoding* based on the extended min-sum (EMS) algorithm [9, 10] is studied. Its assessment in an 800ZR scenario (6.94 bit/symbol) reveals that the complexity is radically decreased whereas the 4D gains in power efficiency are almost entirely exploited.

2. Lattice-Based 4D Constellations

Constellations drawn from the D_4 lattice are more sophisticated than DP-QAM ones: the cardinality can be doubled (cf. 2D projection in Fig. 1) while the minimum distance remains the same and the additional implementation penalty is negligible (*packing gain* [5]). *One additional bit* over DP-QAM is achieved without any significant loss.

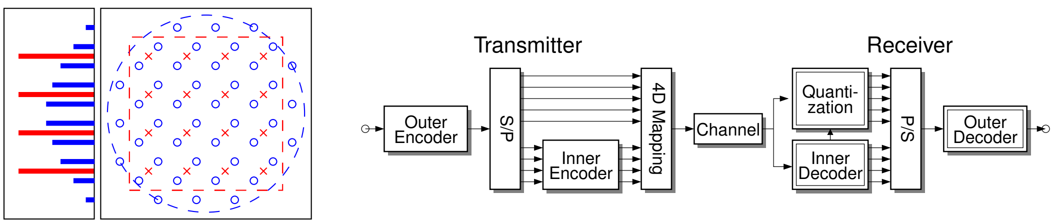

Fig. 1. Left: 2D projections of the 512-ary Welti constellation (blue circles) and DP-16QAM (red crosses). The amplitude distributions per component are shown at the left-hand side. Right: System model of concatenated non-binary FEC assuming a 512-ary Welti constellation (9 bit levels).

An extra *shaping gain* is obtained when the signal points are bounded by a 4D hypersphere instead of a square/hypercube (cf. Fig. 1), known under the name *Welti constellations* [3,5,6]. This shaping gain is enabled by a Gaussian-like amplitude distribution *per component*. Throughout the paper, a 512-ary Welti constellation will be considered and compared to DP-16QAM (256 4D signal points). For the spectral efficiency of 6.94 bit/symbol, the Welti constellation achieves a theoretic packing and shaping gain of 0.79 dB in signal-to-noise ratio (SNR).

3. Concatenated Non-Binary FEC

The MI can be asymptotically achieved by MLC or by jointly protecting all levels with a non-binary code. Since the former results in buffering issues and the latter is far too complex, a mixture has been proposed in [6], cf. Fig. 1 (right): In a concatenated scheme with outer HD and inner SD decoding, 4 bit levels are handled by the inner encoder. Then, all 9 levels are mapped in such a way that a 4D set partitioning is established in which the minimum squared distance grows significantly after 4 levels. Consequently, the bit-error ratio (BER) in the 5 uncoded upper levels is already below the target one for the HD decoder (post-FEC BER below 10^{-15}). At the receiver, only 16-ary decoding is required; in the upper levels, a simple quantization operation is sufficient.

Concatenated *binary* coding has already been specified in the IA 400ZR, particularly realized by an SD-decoded Hamming code and a $(255,239)$ HD-decoded staircase code. Throughout this work, this staircase code is assumed for outer coding with a target BER of $3 \cdot 10^{-3}$ after inner decoding [11]. Beyond that, the post-FEC block length of 2048 symbols (here: used for inner coding), the FEC overhead of 15.3 %, and the symbol rate of 118.2 GBaud are taken from 800ZR specifications [1], resulting in a spectral efficiency of 6.94 bit/symbol for DP-16QAM. The extra bit of the 512-ary Welti constellation is used for additional redundancy in a $(2048,1230)$ non-binary code.

4. Low-Complexity Non-Binary Decoding

The complexity of non-binary BP decoding is dominated by check-node processing, in which convolutions of q-ary probability vectors with asymptotic complexity $\mathcal{O}(q^2)$ are present [8,9], q denoting the size of the field $\mathbb{F}_q$ (here: $q = 16$). Utilizing the fast Fourier transform (FFT), the complexity is slightly reduced to $\mathcal{O}(q\log_2(q))$.

A *significant* complexity reduction is achieved by *message truncation* using the EMS algorithm [9]: The q-ary vectors are truncated to q_t-ary ones, in which only the most likely elements remain. Hence, via q_t, a trade-off between performance and complexity is established, reducing the complexity to $\mathcal{O}(q_t\log_2(q_t))$. Thereby, *multiplications are replaced with comparisons* to obtain the most likely elements (bubble-check algorithm [10]).

Another low-complexity strategy is the use of *ultra-sparse* LDPC codes (constant variable-node degree of 2). Such non-binary codes perform well even for short code lengths [9]. For this work, ultra-sparse binary/non-binary codes with left random part (optimized girth) and right staircase part for systematic encoding have been designed.

5. 800ZR Complexity and BER Performance

The low-complexity non-binary FEC scheme is assessed in terms of an 800ZR scenario. To this end, BICM with DP-16QAM and individual processing per component (2 bit/symbol) is the reference. The binary (4096/3790) ultra-sparse code leads to the desired block length of 2048 symbols (cf. post-FEC block length of OFEC [1]).

Table 1 shows the related numbers of floating-point operations. The FFT-based non-binary approach has an increased computational complexity over binary BP decoding based on log-likelihood ratios (LLRs). Depending on q_t, the EMS algorithm can reduce the number of additions. However, most important is that *multiplications are completely replaced by comparisons*, which are usually much less costly. For EMS with $q_t = 4$, only 96,752 additions are required, whereas for binary decoding, 90,948 multiplications and 32,768 additions are needed.

The BER after the inner decoder is finally assessed by numerical simulations in Fig. 2. To that end, the linear range of the fiber is modeled by the additive white Gaussian noise (AWGN) channel.

Table 1. Computational complexity per iteration (total number of operations for inner variable and check-node processing) assuming BICM (DP-16QAM) as well as non-binary FFT-based BP and EMS decoding (512-ary Welti constellation) of ultra-sparse LDPC codes in an 800ZR scenario.

Operation	Nodes	BICM BP (LLRs)	FFT BP	EMS $q_t = 8$	EMS $q_t = 6$	EMS $q_t = 4$	EMS $q_t = 2$
Multiplication	Variable	–	32,768	–	–	–	–
	Check	90,948	380,224	–	–	–	–
Division	Variable	–	32,768	–	–	–	–
Addition	Variable	32,768	30,720	16,384	12,288	8,192	4,096
	Check	–	262,144	147,600	118,080	88,560	59,040
Comparison	Variable	–	–	49,152	31,764	16,384	4,096
	Check	–	–	354,240	221,400	177,120	88,560

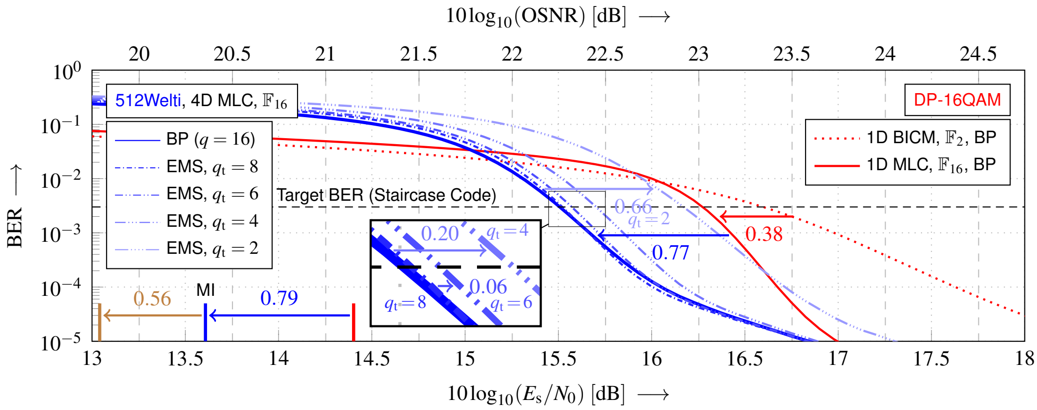

Fig. 2. BER after inner decoding (10 iterations, max(-log) metric) over the SNR (energy/symbol E_s vs. noise-power density N_0) and the OSNR (reference bandwidth 12.5 GHz) given an AWGN channel (2^{15} words). DP-16QAM with 1D BICM and 16-ary coding (1D MLC) as well as 16-ary BP and EMS decoding for the 512-ary Welti constellation. The related MI values (6.94 bit/symbol; vertical lines; Shannon limit: brown) and the target BER for outer coding (horizontal, dashed) are provided.

Again, (1D) BICM with DP-16QAM is used as a reference (red dotted line). In addition, the curve for DP-16QAM with 1D MLC [11] and 1 coded bit level per component (1D set partitioning) is shown, for which the 4 lower levels have jointly been protected by a 16-ary $(2048, 1742)$ code (red solid line). In this way, the well-known significant gain of non-binary over binary BP decoding [9] becomes apparent (here: 0.38 dB at target BER).

Considering the 512-ary Welti constellation with conventional non-binary BP decoding (blue solid line), a gain of about 0.77 dB is obtained over non-binary-coded DP-16QAM. This gain coincides with the theoretic capacities (MI). When applying EMS decoding with $q_t = 8$, the SNR loss is negligible. For $q_t = 6$ and $q_t = 4$, the loss amounts to 0.06 dB and 0.20 dB, respectively. Only if $q_t = 2$ a significant loss is present; however, still with a slight gain over DP-QAM. In summary, EMS decoding with $q_t = 6$ and $q_t = 4$ enables an excellent trade-off between the complexity and the exploitation of the gains obtained by 4D signaling and non-binary coding.

6. Conclusions

Low-complexity inner decoding for concatenated non-binary FEC has been studied. In an 800ZR scenario, more than 1 dB SNR gain over binary LDPC coding and DP-16QAM are possible while multiplications are completely avoided. Due to the excellent trade-off between power efficiency and complexity, this scheme may be suited for future 1600G systems in which DP-64QAM or its 8192-ary 4D Welti equivalent [7] may alternatively be applied.

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