Optimal Nonlinear Spectral Back-Rotation for Discrete Eigenvalue NFT Transmission Systems

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Abstract: We propose back rotating the nonlinear spectral phase by half of the transmission distance as a computationally simple impairment compensation algorithm for discrete eigenvalue NFT transmission systems.

1. Introduction

Nonlinear Frequency Division Multiplexing (NFDM) is a promising theory for addressing the fiber nonlinearity-induced capacity crunch and as an analysis framework for various nonlinear optical systems and phenomena. Using nonlinear Fourier transform (NFT) corresponding to the nonlinear Schrodinger equation (NLSE) that governs a signal propagation in a nonlinear fiber, information can be encoded into the signal's nonlinear spectrum and each nonlinear spectral component undergoes linear evolution (phase rotation) during propagation in a lossless fiber without mutual interference, giving rise to a set of parallel communication channels ideal for the nonlinear fiber in principle [1]. However, the integrability of NLSE is broken as the real-world fiber is always lossy and amplified spontaneous emission (ASE) noise from optical amplifiers also add noise to the nonlinear spectrum, and therefore the signal cannot be recovered by simply back rotating the nonlinear spectrum. This is especially problematic for the discrete NFT system as the discrete eigenvalues are also corrupted by noise and the back rotating the phase spectra will result in large errors. Various signal processing techniques such as linear minimum mean square error (LMMSE) estimator [2], nonlinear filters and machine learning [3] are studied to reduce the noise in eigenvalues and b-coefficients but they are relatively complex.

In this paper, we analytically derived and propose back rotating the received nonlinear spectrum by only *half* of the full propagation distance in practical lossy discrete eigenvalue transmission with inline amplifier noise as a simple and efficient algorithm to improve the SNR of discrete eigenvalue transmission systems. Simulation results show that the proposed algorithm achieves nearly the same performance as LMMSE but significantly reduces computational complexity.

2. Principle of nonlinear spectral back-rotation

In an ideal lossless fiber, the signal propagation is described by NLSE (in normalized form), $jq_z + q_{tt}/2 + |q|^2 q = 0$, where $q = q(z,t)$ denotes the normalized complex envelope of the signal, and the subscripts denotes partial derivatives with respect to z or t. The nonlinear spectrum of $q(z_0, t)$, a signal at position z_0, is obtained by solving the eigenvalue problem, and the nonlinear Fourier coefficients $a(\lambda)$ and $b(\lambda)$ are given by

$$\frac{\partial \boldsymbol{v}}{\partial t} = \begin{pmatrix} -j\lambda & jq(t) \\ jq^*(t) & j\lambda \end{pmatrix} \boldsymbol{v} \qquad \begin{bmatrix} a(\lambda) \\ b(\lambda) \end{bmatrix} = \lim_{t\to+\infty} \begin{bmatrix} v_1 e^{j\lambda t} \\ v_2 e^{-j\lambda t} \end{bmatrix} \; with \; \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \xrightarrow{t\to-\infty} \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{j\lambda t}$$

The continuous and discrete spectral coefficient is defined as $Q_c(\lambda) = b(\lambda)/a(\lambda)$, $Q_d(\lambda_i) = b(\lambda_i)/a'(\lambda_i)$ where λ_i is the solutions of $a(\lambda_i) = 0$, and $\lambda \in \mathbb{R}$, $\lambda_i \in \mathbb{C}^+$, while in this paper, we restrict our discussion in discrete eigenvalue systems. The evolution of b-coefficient, $b(\lambda_i)$ with z is given by $b(\lambda_i, z) = b(\lambda_i, 0)e^{2j\lambda_i^2 z}$, where the eigenvalue(s) λ_i are keep constant during propagation. and we assume information is encoded in the amplitude and phase of b-coefficient. At the receiver, $b(\lambda_i, 0)$ can be simply retrieved by back rotating $b(\lambda_i)$ by $e^{-2j\lambda_i^2 z}$ in the ideal case of lossless and noiseless fiber.

The circularly symmetric complex white Gaussian ASE noise in the time-domain is generally not Gaussian in nonlinear spectrum domain, but the noise in discrete eigenvalues can still be approximated by a conditional Gaussian distribution in the small noise limit [4]. Consider a lumped amplification system and single-eigenvalue discrete NFDM signal with initial eigenvalue $0 + j0.5$, i.e., a 1st-order soliton in the time domain. We assume that at each EDFA (separated by normalized distance L), independent noise is added to α, β and b, and α, β and b are free from noise during the propagation in the next fiber span. Their values right after the i^{th} EDFA are denoted by α_i, β_i and b_i respectively as in Fig 1.

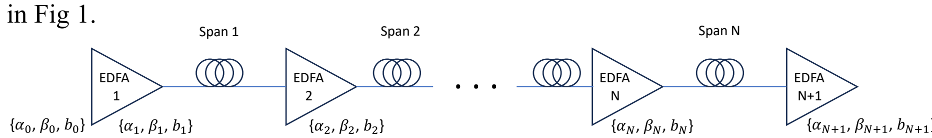

Fig.1 Schematic diagram of the transmission link. Gaussian noise is independently added to λ and b after each EDFA.

During propagation, the b-coefficient gradually deviates from its original value due to its own noise accumulation and also due to the noisy eigenvalue. Mathematically, $b_{i+1} = b_i e^{-2j(\alpha_i + j\beta_i)^2 L} + n_{i+1}$, so $|b_{i+1}| \approx |b_i| e^{4\alpha_i \beta_i L} \approx |b_i| e^{2\alpha_i L}$ and $\angle b_{i+1} \approx \angle b_i + 2(\beta_i^2 - \alpha_i^2)L \approx \angle b_i + 2\beta_i^2 L$ in the small noise case, as shown in Fig. 2. We note that the deviations of α and β at each EDFA is relatively small, which makes the deviation of α contributes much more significantly to the deviation of $|b_i|$, while the noise in β contribute much more to the deviations of $\angle b_i$, resulting in the approximation above. It is more common to analyze the logarithm of the b-coefficient, so $\ln|b_{N+1}| = \ln|b_N| + 2\alpha_N L = \cdots = \ln|b_0| + 2L\sum_{i=1}^{N} \alpha_i$ and similarly, $\angle b_{N+1} = \angle b_0 + 2L\sum_{i=1}^{N} \beta_i^2$. Without loss of generality, we set the initial b-coefficient, $b_0 = 1$. We also assume the noise added to either α or β at each EDFA is identical independently Gaussian distributed and ignore the noise in the b-coefficient itself hereafter (while it is still drawn in Fig. 2). That is to say, α_i and β_i are roughly a “discrete” version of Wiener processes. More complex models can be found in [5-7], which argued that the noise in the discrete eigenvalues is generally not Gaussian.

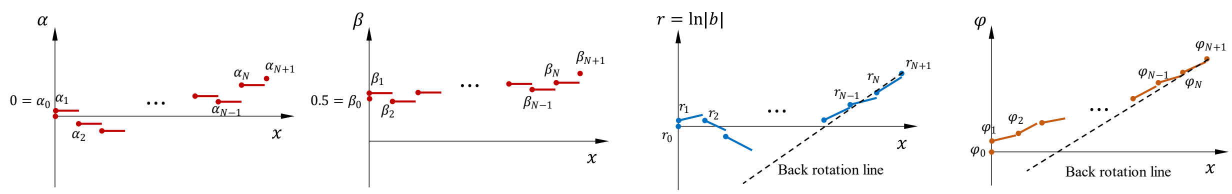

Fig. 2 Random evolution of λ_i and b_i due to the ASE noise added by each EDFA, and the x-axis is normalized to L, $x = z/L$.

After transmitted through a N-span system with N+1 EDFA, the signal's noisy eigenvalue $\alpha_{N+1} + j\beta_{N+1}$ and b-coefficient b_{N+1} are obtained through NFT operation, and we back rotate the noisy b-coefficient by their corresponding received noisy eigenvalues. This is graphically shown as the straight dash line in Fig. 2 starting from point (N, b_{N+1}) with a slope of $2L\alpha_{N+1}$ or $2L\beta_{N+1}^2$. Note that in the noiseless case, the evolution trace of b-coefficient and its back rotation line are exactly the same, so a perfect recovery of b-coefficient can be achieved. However, in the noisy case, eigenvalue evolves in a random-walk manner and the b-coefficient evolution trace is approximately the summation of the random walk. In this case, the evolution trace and back rotation line will never be the same. More specifically, the back rotation line for $\ln|b_{N+1}|$ is $r(x) = 2L\alpha_{N+1}(x - N) + r_{N+1}$, where $r(x) = \ln|b(x)|$, $r_{N+1} = \ln|b_{N+1}|$ and $x = z/L$ for clear notation. Then,

$$\mathrm{Var}[r(x)] = 4L^2(x-N)^2\mathrm{Var}(\alpha_{N+1}) + \mathrm{Var}(r_{N+1}) + 4L(x-N)\mathrm{cov}(r_{N+1}, \alpha_{N+1}) = 4L^2\sigma_\alpha^2(N+1)\left[x(x-N) + \frac{N(2N+1)}{6}\right] \quad (1)$$

where $\mathrm{E}[\alpha_i \alpha_N] = i\sigma_\alpha^2$ has been used, and σ_α^2 is introduced as the variance of the noise added to α at each EDFA, which can be determined by simulation or experiment. The $N(N+1)(2N+1) \approx 2N^3$ term derived from the variance of r_{N+1}, and note this cube dependence is consistent with the Gordon-Haus effect [8]. To minimize Var[r(x)], we let $\frac{d}{dx}\mathrm{Var}[r(x)] = 0$ and obtain $x_0 = N/2$. Similarly, the back rotation line for $\angle b_N$ is $\varphi(x) = 2L\beta_{N+1}^2(x - N) + \varphi_{N+1}$, where φ represents $\angle b$. Thus,

$$\mathrm{Var}[\varphi(x)] = 4L^2(x-N)^2\mathrm{Var}(\beta_{N+1}^2) + \mathrm{Var}(\varphi_{N+1}) + 4L(x-N)\mathrm{cov}(\varphi_{N+1}, \beta_{N+1}^2) \approx 4L^2\sigma_\beta^2(N+1)\left[x(x-N) + \frac{N(2N+1)}{6}\right] \quad (2)$$

where σ_β^2 stands for the variance of noise to β at each EDFA. We have used $\mathrm{Var}(\beta_{N+1}^2) = \mathrm{Var}[(w_{N+1} + 0.5)^2] = \mathrm{Var}(w_{N+1}^2) + \mathrm{Var}(w_{N+1}) + 2\mathrm{cov}(w_{N+1}, w_{N+1}^2) = 2\sigma_\beta^4(N+1)^2 + \sigma_\beta^2(N+1)$, $\mathrm{cov}(\varphi_{N+1}, \beta_{N+1}^2) = 2L\sum_{i=1}^{N}\mathrm{cov}(\beta_i^2, \beta_{N+1}^2) = 2L\sum_{i=1}^{N}\{\mathrm{cov}(w_i^2, w_{N+1}^2) + \mathrm{E}[w_i w_{N+1}]\} = 2L\sum_{i=1}^{N}\{2i^2\sigma_\beta^4 + i\sigma_\beta^2\}$. The σ_β^4 terms were then neglected as $\sigma_\beta^2 \ll 1$ in practice, and we obtained the final expression. Let $\frac{d}{dx}\mathrm{Var}[\varphi(x)] = 0$, and we also obtain $x_0 = N/2$.

In short, both the variance of the amplitude and angle of the noisy b-coefficient is minimized when the back rotation length is half of the full transmission distance. Such result has some resemblance to the optimal scale factor for nonlinear phase noise compensation in chromatic dispersion-free scenario [9]. Note that when a full back rotation of the nonlinear spectral phase happens ($x = 0$), both the variance of $\ln|b|$ and φ will be as large as those at the receiver, hence a full back rotation does *not* help to reduce the noise on b-coefficients at all.

3. Results

We conducted simulations on a 40×50 km EDFA system with fiber loss of 0.2dB/km. 1024 path-averaged 1st-order soliton pulses with 16APSK-modulated b-coefficients were used to estimate the variance of the absolute value, angle, and SNR of the b-coefficients during the forward evolution and back rotation process. The ratio of soliton's FWHM to symbol interval was set to 8.8 to avoid interactions between the adjacent solitons so the effect of noise is isolated. We studied the evolution of eigenvalues and b-coefficients of the soliton pulses every 10 km, and for back rotation, we evaluated the b-coefficients every 100 km. Fig. 3a shows the evolution of α and the histogram of its "jumps" at each EDFA. It can be seen that α is roughly constant between EDFAs but randomly jumps after each EDFA. Fig. 3b shows the evolution and back rotation traces of $\ln|b|$ and $\angle b$, where the red and black dots represent the difference between the current b-coefficient to its initial value of each *random* pulse during forward evolution and back rotation. The variance indeed reaches its minimum and SNR reach the maximum near the midpoint of the full transmission distance and the SNR for full back rotation is similar to that at Rx, in agreement with the theoretical analysis.

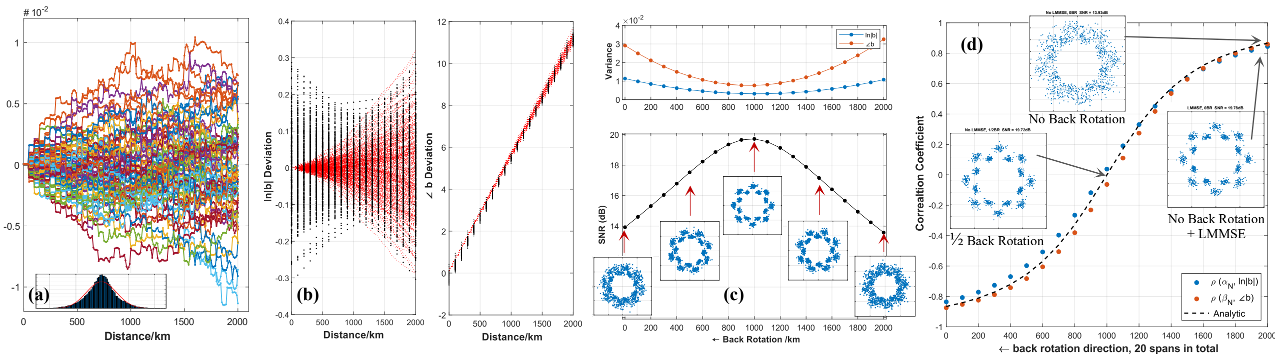

Fig. 3 (a) Evolution of α and the histogram of α noise at each EDFA (only 1/10 of the traces are shown for visual clarity); (b)Evolution and back rotation of $\ln|b|$ and $\angle b$; (c) Variance of $\ln|b|$, $\angle b$ and the SNR of b-coefficient during back rotation; (d) Simulated and analytic correlation coefficients between the b and λ during back rotation.

We also found that the SNR of the signal using the proposed method are similar to that by the LMMSE method [2] and we found that further applying LMMSE to the half back rotated signal does not improve the SNR much. This motivates us to investigate the correlation between the b-coefficient and the eigenvalue along the back rotation process, since LMMSE essentially makes use of this correlation to denoise b-coefficient. With the noise model described above, the correlation coefficient between $r(x)$, α_{N+1} and $\varphi(x)$, β_{N+1} can be obtained from the derivations above as

$$\rho(\varphi(x), \beta_{N+1}) \approx \rho(r(x), \alpha_{N+1}) = (x - N/2)/\sqrt{x(x-N) + N(N+1)(2N+1)/6} \quad (3)$$

The analytical expression matches the simulated correlation coefficients quite well as shown in Fig. 3(d), and clearly the correlation coefficient reduces to zero with half back rotation and reaches to its maximum of $\pm\sqrt{3}/2$ with no or full back rotation.

4. Conclusions

We propose back rotating the received noisy b-coefficient by half of the transmission distance with the received noisy eigenvalues. The effectiveness of half back rotation was derived analytically and shown numerically, and it gives the same performance compared to LMMSE but with much less calculation complexity.

5. Acknowledgement

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5. References

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