Full Spectrum b-Modulation of Time-Limited Signals Using Linear Programming

Sander Wahls
Delft Center for Systems and Control, TU Delft, The Netherlands
s.wahls@tudelft.nl

Abstract: We present the first method for the joint modulation of the continuous and the discrete nonlinear Fourier spectrum of finite duration signals.

1. Introduction

The nonlinear Fourier transform (NFT) linearizes the ideal nonlinear fiber channel, which has led to much interest in the area of fiber-optic communication systems in the last decade [1]. The spectral efficiencies of early systems were very low, but have been steadily increasing. Recently, a value of 5.51 bits/sec/Hz over a 960 km link has been demonstrated in simulations [2], which is only about a factor two away from conventional systems. The design of NFT-based transceivers is complicated by the fact the NFT consists of two parts: a continuous spectrum for radiation components, and a discrete spectrum for solitonic components. The high spectral efficiency in [2] was achieved by modulating the continuous spectrum only, using a variant of b-modulation [3]. The advantage of b-modulation is that it offers control over the temporal support of the generated pulses, a quality earlier approaches were lacking. Furthermore, it turned out to be more robust to noise. One the other hand, it has been shown that b-modulation can suffer from power constraints [4]. While there have also been demonstrations of NFT-based systems that use only the discrete spectrum (multi-soliton transceivers) or combine a few solitons with a continuous spectrum, these approaches have not been reported to achieve high spectral efficiencies so far.

Inspired by the observation that the NFTs of conventionally modulated signals are typically strongly dominated by solitonic components [5], it was recently proposed to improve the pulse shapes of multi-solitons by adding a suitable continuous spectrum that shorten them to a desired finite duration [6]. The advantage of (multi-)soliton shortening over simple pulse truncation is that does not perturb the discrete spectrum. As a proof of concept, it was shown that soliton shortening increased the spectral efficiency of a two-soliton on-off-keying transceiver by 40% over a 960 km link. In this paper, we present a new method for full spectrum b-modulation (FSbMOD) that unifies classic b-modulation for the continuous spectrum [3] with (multi-)soliton shortening for discrete spectra [6]. *FSbMOD is the first method that can jointly modulate the continuous and discrete parts of the nonlinear spectrum under a finite duration constraint.* The proposed algorithm therefore offers control over the temporal support which conventional full spectrum modulation is missing. Since both parts of the spectrum are utilized, FSbMOD is not limited in power like classical b-modulation. The FSbMOD algorithm is based on a systematic optimization approach and can thus be expected to succeed if solutions exist (and tuned properly). The soliton-shortening methods in [6] in contrast were ad-hoc and could fail even though solutions exist.

2. Theoretical Background of the Nonlinear Fourier Transform (NFT) (see, e.g., [1,6,7])

Using path-averaging and normalization, the evolution of a periodically amplified pulse in an optical fiber link is approximated by the nonlinear Schroedinger equation (NSE) $\mathrm{j}q_z + q_{tt} + 2q|q|^2 = 0$. Here, $q(z,t)$ is the normalized complex envelope of the pulse, z denotes normalized position, t normalized time, $\mathrm{j} = \sqrt{-1}$, and the subscripts indicate partial derivatives. The NFT of $q(z_0,t)$ with z_0 fixed is defined using the Zakharov-Shabat problem

$$\frac{d}{dt}\begin{bmatrix}\phi_1(t;\lambda)\\ \phi_2(t;\lambda)\end{bmatrix} = \begin{bmatrix}-\mathrm{j}\lambda & q(z_0,t)\\ -q^*(z_0,t) & \mathrm{j}\lambda\end{bmatrix}\begin{bmatrix}\phi_1(t;\lambda)\\ \phi_2(t;\lambda)\end{bmatrix}, \quad \begin{bmatrix}1\\0\end{bmatrix} \overset{t\to-\infty}{\longleftarrow} \begin{bmatrix}\mathrm{e}^{\mathrm{j}\lambda t}\phi_1(t;\lambda)\\ \mathrm{e}^{-\mathrm{j}\lambda t}\phi_2(t;\lambda)\end{bmatrix} \overset{t\to+\infty}{\longrightarrow} \begin{bmatrix}a(\lambda)\\ b(\lambda)\end{bmatrix},$$

where λ is a complex parameter and $(\cdot)^*$ indicates the complex conjugate. The NFT of $q(z_0,t)$ consists of the continuous spectrum $b(\xi), \xi \in \mathbb{R}$, and the discrete spectrum $\{\lambda_k, b_k\}_{k=1}^{K}$, where the eigenvalues λ_k are the solutions to $a(\lambda) = 0$ in $\Im(\lambda) > 0$ and $b_k = b(\lambda_k)$. Here, $\Im(\cdot)$ indicates the imaginary part. If $q(z_0,t)$ is time-limited, then

$$a(\lambda)a^*(\lambda^*) + b(\lambda)b^*(\lambda^*) = 1\ \forall\lambda \in \mathbb{C} \quad \Longrightarrow \quad b(\lambda_k)b^*(\lambda_k^*) = 1, \quad |b(\xi)|^2 = 1 - |a(\xi)|^2 < 1\ \forall\xi \in \mathbb{R}. \tag{1}$$

The evolution of the nonlinear spectrum w.r.t. the normalized NSE is trivial with $\lambda_k(z) = \lambda_k(0)$ and $b(\lambda;z) = b(\lambda;0)\exp(4j\lambda^2 z)$. To generate time-limited pulses, ones considers bandlimited continuous spectra of the form

$$b(\lambda) = \frac{1}{2\pi}\int_{-2T}^{2T} B(\tau)\mathrm{e}^{\mathrm{j}\lambda\tau}d\tau, \quad B(\tau) \text{ absolutely integrable.} \tag{2}$$

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 716669).

If $|b(\xi)| < 1$ for all real ξ and the discrete spectrum satisfies $b(\lambda_k) = b_k$, $b(\lambda_k^*) = 1/b_k^*$, then the inverse NFT results in a time domain signal $q(z_0,t)$ that is time-limited in the sense that $q(z_0,t) = 0$ for $|t| > T$.

3. Full Spectrum b-Modulation (FSbMOD) Algorithm

We now describe the FSbMOD algorithm. The algorithm contains both classic b-modulation and soliton shortening as special cases. The problem will be formulated as an interpolation problem that is then solved using linear programming. The approximation errors in the formulation and the linear program can be made arbitrarily small.

Problem Statement The goal is to find – if it exists – a time-limited pulse of duration $2T$ with a desired nonlinear Fourier spectrum. The algorithm therefore determines a $b(\lambda)$ as in (2) that satisfies the interpolation conditions

$$\underbrace{\underline{r}_n \le \Re(b(\xi_n)) \le \overline{r}_n, \quad \underline{i}_n \le \Im(b(\xi_n)) \le \overline{i}_n,}_{\text{continuous spectrum}} \quad \underbrace{b(\lambda_k) = b_k}_{\text{discrete spectrum}}, \quad \underbrace{b(\lambda_k^*) = 1/b_k^*, \quad |b(\nu_l)| < 1-\varepsilon}_{\text{required because of (1)}} \quad \forall n,k,l, \tag{3}$$

where the desired pulse duration $2T$, the interpolation nodes ξ_n and λ_k, the bounds $\underline{r}_n$, $\overline{r}_n$, $\underline{i}_n$, $\overline{i}_n$ and the values b_k are specified by the user; $\Re(\cdot)$ denotes the real part. Note that it is possible to enforce equality constraints on the continuous spectrum by setting $\underline{r}_n = \overline{r}_n$ and $\underline{i}_n = \overline{i}_n$, but it can be advantageous to relax them. It is not possible to relax the constraints on the discrete spectrum since that destroys the symmetry $b(\lambda_k^*) = 1/b^*(\lambda_k)$ required by (1). The purpose of the real grid points ν_l is to approximate the condition $|b(\xi)| < 1$, which is also required by (1).

Description of the Algorithm To find a $b(\lambda)$ that satisfies both (2) and (3) numerically, we exploit that bandlimited functions can be expressed using generalized sampling series

$$b(\xi) = \sum_{r=-R}^{R} c_r \psi(\xi - Wr), \quad \psi(\xi) = \frac{1}{W}\frac{\sin(\pi\xi/W)}{\pi\xi/W}\frac{\cos(\pi\beta\xi/W)}{1-(2\beta\xi/W)^2} \text{ for } \xi \notin \{0, \pm W/(2\beta)\},\ W = \pi\frac{1+\beta}{2T}, \tag{4}$$

where $c_r = b(Wr)$, $R = \infty$, $W > 0$ and $0 \le \beta \le 1$. To make the problem computationally feasible, we consider only finite values of R in (4). The task of the algorithm now reduces to finding coefficients c_r such that (3) is satisfied by (4). Note that $\psi(\xi)$ is the frequency response of a raised cosine filter, so that $T = \pi(1+\beta)/(2W)$ in (2). Let us now introduce the evaluation matrix $\mathbf{V}(z_{i_1},\dots,z_{i_2}) = [\psi(z_i - Wr)]_{i=i_1,r=-R}^{i_2,R}$ for arbitrary complex interpolation nodes z_i. Furthermore, let $\mathbf{v}(y_1,\dots,y_i)$ denote the column vector obtained by stacking arbitrary values $y_1,\dots,y_i$. The first interpolation condition in (3) can then be written as the real-valued linear equation system

$$\begin{aligned}\mathbf{v}(\underline{r}_1,\dots,\underline{r}_N,\underline{i}_1,\dots,\underline{i}_N) &\le \mathsf{C}[\mathbf{V}(\xi_1,\dots,\xi_N)]\mathsf{c}[\mathbf{c}] \\ &\le \mathbf{v}(\overline{r}_1,\dots,\overline{r}_N,\overline{i}_1,\dots,\overline{i}_N)\end{aligned}, \text{ where } \mathsf{C}[\cdot] := \begin{bmatrix}\Re(\cdot) & -\Im(\cdot)\\ \Im(\cdot) & \Re(\cdot)\end{bmatrix} \text{ and } \mathsf{c}[\cdot] := \begin{bmatrix}\Re(\cdot)\\ \Im(\cdot)\end{bmatrix}. \tag{5}$$

We can similarly express the second and third interpolation condition in (3) as $\mathsf{C}[\mathbf{V}(\lambda_1,\dots,\lambda_K)]\mathsf{c}[\mathbf{c}] = \mathsf{c}[\mathbf{v}(b_1,\dots,b_K)]$ and $\mathsf{C}[\mathbf{V}(\lambda_1^*,\dots,\lambda_K^*)]\mathsf{c}[\mathbf{c}] = \mathsf{c}[\mathbf{v}(1/b_1^*,\dots,1/b_K^*)]$. The fourth condition in (3) contains complex absolute values. We can approximate it arbitrarily well by the S real-valued systems of conditions [8]

$$\begin{bmatrix}\cos(\theta_s) & \sin(\theta_s)\end{bmatrix} \mathsf{C}[\mathbf{V}(\nu_1,\dots,\nu_n)]\mathsf{c}[\mathbf{c}] \le \begin{bmatrix}1 & \dots & 1\end{bmatrix}^T, \quad \theta_s = 2\pi(s-1)/S, \quad s = 1,\dots,S. \tag{6}$$

Using that $\mathbf{y} \le \mathbf{Bx} \Leftrightarrow -\mathbf{Bx} \le -\mathbf{y}$ and $\mathbf{Bx} = \mathbf{y} \Leftrightarrow \mathbf{y} \le \mathbf{Bx} \le \mathbf{y}$ for arbitrary matrices $\mathbf{B}$ and vectors $\mathbf{x},\mathbf{y}$ with compatible dimensions, we can combine the condition (5) with the others in one large system of inequalities of the form $\mathbf{A}\mathsf{c}[\mathbf{c}] \le \mathbf{b}$. This is a linear programming problem that can be solved for $\mathsf{c}[\mathbf{c}]$ with standard solvers.

Summary The full spectrum b-modulation algorithm works as follows. The user specifies the desired nonlinear spectrum by providing the nodes ξ_n and λ_k, bounds $\underline{r}_n$, $\overline{r}_n$, $\underline{i}_n$ and $\overline{i}_n$ as well as the values b_k for (3). The desired duration $2T$ and the parameters ν_l, ε in (3), R in (4) and S in (6) that control the quality of approximations also have to be provided. The linear program $\mathbf{A}\mathsf{c}[\mathbf{c}] \le \mathbf{b}$ described above is then solved – if it is solvable – for $\mathsf{c}[\mathbf{c}]$. Together with the specified discrete spectrum $\{\lambda_k, b_k\}$, the resulting continuous $b(\xi)$ from (4) is provided to an inverse NFT algorithm, which computes the desired time-domain signal $q(z_0,t)$. This signal will be of finite duration $2T$.

4. Numerical Example

The proposed algorithm is the first that can modulate the complete nonlinear Fourier spectrum under a finite duration constraint. We now compare it with conventional full spectrum modulation (CFSMOD), which was implemented by using the FSbMOD algorithm without the discrete spectrum constraints so that the continuous spectrum would not be matched to it. The FSbMOD algorithm is run with the carrier parameters $\beta = 0.1$, $W = 0.4$, $R = 10$, and $S = 24$. The discrete spectrum is $\lambda_k = 0.5j + k - 2$, $k = 1,2,3$. The b_k are QPSK-modulated b_k with $|b_k| = e^{\Re(\lambda_k)}$. The continuous spectrum nodes are $\xi_n = 2W(n-5)$, $n = 1,\dots,9$. The continuous spectrum

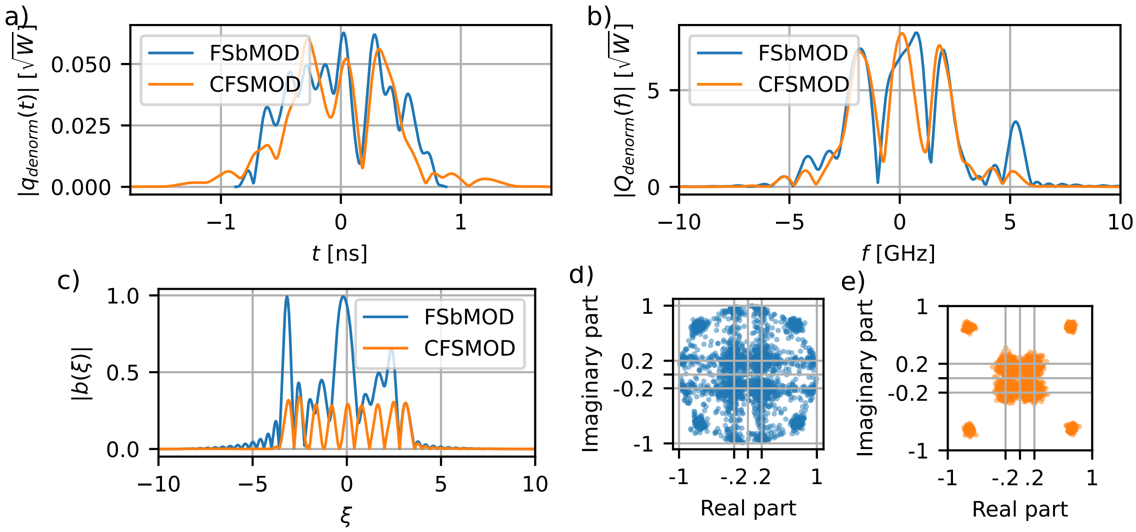

Fig. 1. Numerical example. The link parameters used for denormalization and simulation are $\alpha = 0.2$ dB/km, $\beta = -21.5 \times 10^{-27}$ s^2/m, $\gamma = 1.3 \times 10^{-3}$ 1/(Wm), lumped amplification, 5dB noise figure.

was also QPSK modulated, but in a relaxed form. We only require the algorithm to achieve $\Re(b(\xi_n)) \gtrless 0.2$ and $\Im(b(\xi_n)) \gtrless 0.2$. This was done because the problem became infeasible when equality constraints were used. For the same reason, the spacing of the ξ_n is $2W$ instead of W. The ν_l are a equidistant grid on $[-(2R+1)W, (2R+1)W]$ with spacing $W/16$. We set $\varepsilon = 0.01$. Fig. 1a shows an exemplary denormalized time domain signal generated using the proposed FSbMOD and using CFSMOD. FSbMOD has clearly shortened the signal to a finite duration. Fig. 1b shows the linear Fourier spectra, while Fig. 1c contains the corresponding continuous nonlinear spectra. The nonlinear spectrum of FSbMOD is stronger than that of CFSMOD since it is also used to shorten the signal. To compare FSbMOD and CFSMOD, the transmission of a train of 250 signals was simulated over a 12×80km link. The signals were pre-compensated by removing half of the phase-shift the NFT incurs at the transmitter (TX). As this widens the signals, the pulses were therefore truncated to different durations $2Ts$, where $s = 0.9, 1, 1.1, 1.2, 1.3, 1.4$ is a scaling factor. Similarly, the TX/RX bandwidth was swept from 6 to 13.5GHz in steps of 500MHz. The constellation diagrams for the best performing configurations [i.e., highest number of bits/s/Hz with a bit error ratio (BER) ≤ 0.02] are shown in the Figs. 1d (FSbMOD) and 1e (CFSMOD). The inner and outer points correspond to the continuous and discrete spectrum, respectively. FSbMOD exploits the relaxed constraint and puts the continuous spectrum at larger values to shorten the signal. The best FSbMOD configuration achieved 0.99 bits/s/Hz at a BER of 0.013, the best CFSMOD achieved 0.96 bits/s/Hz at a BER of 0.0075.

5. Conclusion

We presented the first method for the joint modulation of the continuous and discrete nonlinear spectrum that offers control over the duration of the generated pulses. We demonstrated joint modulation in a numerical example, where the pulse was clearly shortened to a finite duration. However, this did not translate into a higher spectral efficiency. It is not yet clear if that is because of the specific setup considered in the example. The proposed method is very versatile due to its optimization approach and could also be used in other ways (e.g., for power control with conventional b-modulation, modulation of $b(\lambda)$ with λ not an eigenvalue, ...). It furthermore provides a systematic approach to soliton shortening, which has already been shown to be beneficial for soliton on-off keying in [6].

References

1. S. K. Turitsyn et al., "Nonlinear Fourier transform for optical data processing and transmission: advances and perspectives," Optica **4**, 307–322 (2017).
2. Q. Zhang and F. R. Kschischang, "Correlation-Aided Nonlinear Spectrum Detection," IEEE JLT **39**, 4923–4931 (2021).
3. S. Wahls, "Generation of time-limited signals in the nonlinear Fourier domain via b-modulation," in *Proc. ECOC* (2017)
4. S. Chimmalgi and S. Wahls, "Bounds on the transmit power of b-modulated NFDM systems in anomalous dispersion fiber," Entropy **22**, 639 (2020).
5. E. V. Sedov et al., "Soliton content in the standard optical OFDM signal," Optics Lett. **43**, 5985–5988 (2018).
6. S. Wahls, "Shortening Solitons for Fiber-Optic Transmission," in *Proc. Int. Symp. Wireless Commun. Syst.* (2021).
7. D. Shepelsky et al., "Nonlinear Fourier spectrum characterization of time-limited signals," IEEE Trans. Commun. **68**, 3024–3032 (2020).
8. R. L. Streit, "Solution of Systems of Complex Linear Equations in the l_∞ Norm with Constraints on the Unknowns," SIAM J. Sci. Stat. Comput. **7**, 132–149 (1986).