Nonlinear Interference Noise of Constant-Composition Codes

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Abstract *A time-domain perturbation model of the nonlinear Schrödinger equation is used to explain (a) why constant-composition codes offer an improvement in signal-to-noise ratio compared with independent and uniform selection of constellation points and (b) why similar gains are obtained using carrier recovery algorithms.*

Introduction

Recent studies of probabilistic shaping for fiber optic systems have reported that a considerable gain in signal-to-noise ratio (SNR) can be achieved in the presence of Kerr nonlinearity using constant-composition (CC) distribution matchers of short blocklength, but that the gain reduces (or is absent) at longer blocklength[1]–[3]. In one study[4], the effect of a CC distribution matcher on the induced nonlinear interference is attributed to a limited concentration of identical symbols. In another work[5], the temporal energy behavior of symbol sequences is studied, and a new metric, called energy dispersion index, is proposed to predict the impact of blocklength of the CC distribution matcher on the effective SNR.

While probabilistic shaping with finite-length CC codebooks provides demonstrable improvements in effective SNR in the absence of phase-tracking algorithms at the receiver, it is known that the gain in generalized mutual information obtained by probabilistic amplitude shaping via enumerative sphere shaping is about the same as the gain of typical carrier phase recovery algorithms[6]. In particular, no additional shaping gain is observed when a carrier phase recovery module is in place, which is the case in all practical systems. This observation raises the question of whether or not the SNR gain of short-length distribution matchers is of practical importance.

In this paper, we provide a theoretical justification for the superior SNR of short CC codes. Our work is based on a first order perturbation model of data transmission over optical fibers[7]. We verify that the dominant perturbation coefficients are slowly-varying functions of the time-separation of symbols in neighboring channels. We also explain why carrier phase recovery algorithms can provide similar gains.

Channel Model

Noiseless propagation of an optical signal over a single mode fiber without attenuation is described by the nonlinear Schrödinger (NLS) equation[8]

$$\frac{\partial Q(\tau,l)}{\partial l} = -i\frac{\beta_2}{2}\frac{\partial^2 Q(\tau,l)}{\partial \tau^2} + i\gamma\left|Q(\tau,l)\right|^2 Q(\tau,l).$$

In this equation, the constant β_2 is the chromatic dispersion coefficient and γ is the nonlinearity coefficient. The assumption of having no attenuation (and hence no amplifier noise) is merely to be able to isolate the nonlinear signal–signal interactions in the following sections. A more realistic loss profile or lumped amplification, along with higher order dispersive effects, can also be considered.

Model of Nonlinear Interference

Consider a WDM system with $2M+1$ co-propagating channels using unit-energy $\operatorname{sinc}(\cdot)$ pulse shaping in all channels. The signal constellation used by channel k is $A_k \subset \mathbb{C}$. The symbol sent over channel k at time jT is denoted as $a_{k,j} \in A_k$. The launched signal, therefore, is

$$q(0,t) = \sum_{k=-M}^{M}\sum_{j=-\infty}^{\infty}\frac{a_{k,j}}{\sqrt{T}}\operatorname{sinc}\left(\frac{t-jT}{T}\right)e^{i2\pi kBt},$$

where T^{-1} is the baud rate and $B = T^{-1}$ is the channel spacing.

The channel of interest is the middle channel indexed by $k=0$ and the symbol of interest is the one indexed by $j=0$, that is, $a_{0,0}$. Signal detection for the channel of interest is done using a matched filter, i.e., a $\operatorname{sinc}(\cdot)$ function dispersively propagated to distance z. The output of the matched filter is $\hat{a}_{0,0} = a_{0,0} + \Delta$, where Δ represents the noise induced due to nonlinear signal–signal interactions. The most impor-

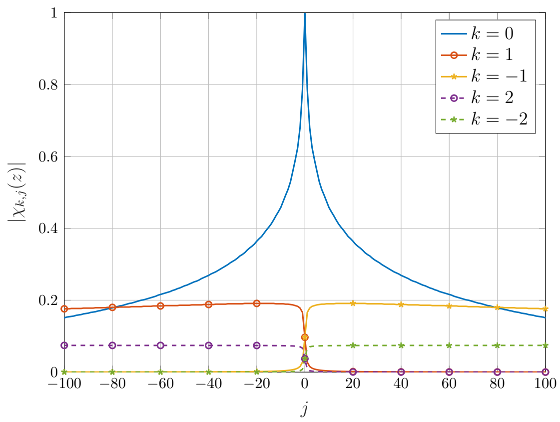

Fig. 1: The normalized perturbation coefficients for a fiber of length 2000 km with $B = 50$ GHz.

tant components of such interactions are the self-phase modulation (SPM) and cross-phase modulation (XPM). In principle, SPM can be undone by equalizing the channel of interest. Consequently, XPM is considered to be the dominant nonlinear interference noise in a WDM system caused by nonlinear signal–signal interactions. Following the first order perturbation analysis of Mecozzi and Essiambre[7], XPM can be approximated by

$$\Delta_{\text{XPM}} \approx 2a_{0,0} \sum_{k\neq 0} \sum_{j} |a_{k,j}|^2 \chi_{k,j}(z), \tag{1}$$

where $\chi_{k,j}(z)$ represents the perturbation coefficients. The normalized absolute value[1] of $\chi_{k,j}(z)$ for a fiber of length 2000 km is shown in Fig. 1. The channel spacing is $B = 50$ GHz. In particular, Fig. 1 confirms that, as long as j is not close to zero, the XPM coefficients are indeed slowly-varying functions of j.

XPM of Constant-Composition Codes

In this section, we consider a WDM system in which all channels use a CC code of length m, in which each codeword consists of m distinct symbols. We assume that the symbols of each codeword are transmitted consecutively, without interleaving; thus if the l^{th} codeword for transmission over channel k is

$$(a_{k,lm}, a_{k,lm+1}, a_{k,lm+2}, \ldots, a_{k,lm+m-1}),$$

the symbols that are sent on channel k from time index $j = lm$ to time index $j = lm + m - 1$ are

$$a_{k,lm}, a_{k,lm+1}, a_{k,lm+2}, \ldots, a_{k,lm+m-1}.$$

We now use the fact that the coefficients $\chi_{k,j}(z)$ are slowly varying with j. For example, for the parameters used in Fig. 1 we have $\chi_{k,j} \approx \chi_{k,h}$ if $|j| > 10$, $|h| > 10$ and $|j - h|$ is not too large. As long as the blocklength m of the CC code used by all channels is small enough so that the above approximation is justified, the XPM term in (1) can be written as

$$\begin{aligned}\Delta_{\text{XPM}} \approx & 2a_{0,0} \sum_{k\neq 0} \sum_{j=-m}^{m-1} |a_{k,j}|^2 \chi_{k,j}(z) \\ & + 2a_{0,0} \sum_{k\neq 0} \sum_{l=1}^{\infty} \chi_{k,lm}(z) \sum_{j=0}^{m-1} |a_{k,lm+j}|^2 \\ & + 2a_{0,0} \sum_{k\neq 0} \sum_{l=-\infty}^{-2} \chi_{k,lm}(z) \sum_{j=0}^{m-1} |a_{k,lm+j}|^2.\end{aligned} \tag{2}$$

If we denote the *energy* of each of the codewords in the CC code used by E, the XPM term becomes

$$\begin{aligned}\Delta_{\text{XPM}} \approx & 2a_{0,0} \sum_{k\neq 0} \sum_{j=-m}^{m-1} |a_{k,j}|^2 \chi_{k,j}(z) \\ & + 2a_{0,0} E \sum_{k\neq 0} \sum_{l=1}^{\infty} \chi_{k,lm}(z) \\ & + 2a_{0,0} E \sum_{k\neq 0} \sum_{l=-\infty}^{-2} \chi_{k,lm}(z).\end{aligned} \tag{3}$$

Notice that the second and third summations in (3) are deterministic. In other words, the most important XPM terms are those that capture the nonlinear interaction of the symbol of interest with the symbols in the neighboring channels that are closest in time to the symbol of interest at the beginning of the fiber. Thus, as long as the blocklength of the CC code is not too large, the effect of most of the XPM terms in (1) is deterministic. The XPM uncertainty, therefore, is limited to the collision of the symbols that are transmitted almost concurrently. Because of this limited XPM uncertainty, the overall observed SNR will be higher than the case of independent and uniformly distributed (IUD) symbol selection.

While CC codes are expected to reduce the uncertainty of XPM, we may at the same time observe from our model that in general the XPM term induced in detection of $a_{0,j}$ and $a_{0,h}$ is almost the same, provided that $|j - h|$ is not too large. In other words, we expect the effect of XPM on nearby symbols to be slowly varying in time. This observation, together with the fact that XPM mostly affects the *phase* of the detected

[1]Since $\chi_{k,j}(z)$ is purely imaginary, knowing the absolute value is enough to know the coefficient.

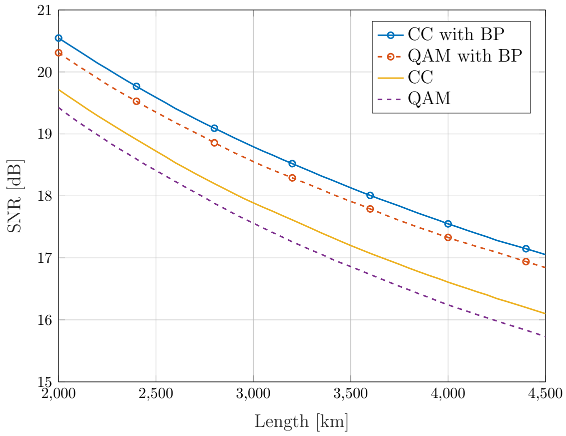

Fig. 2: SNR for a CC code of blocklength 171 and IUD transmission using 64-QAM. Matched-filtering with and without back-propagation (BP) are considered.

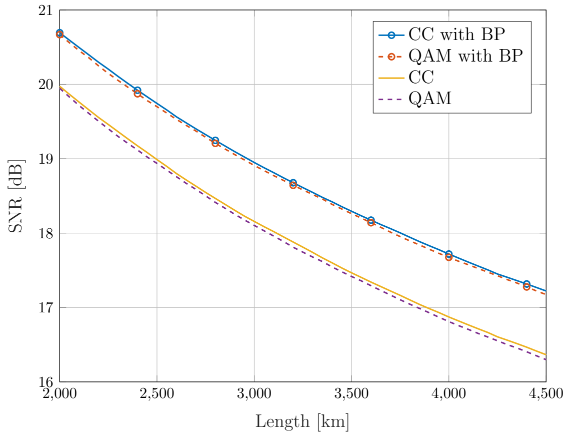

Fig. 3: SNR for a CC code of blocklength 171 and IUD transmission using 64-QAM versus fiber. Matched-filtering with and without back-propagation (BP) are considered.

symbols, suggest that much of the XPM-induced nonlinear interference may be undone by use of phase-tracking algorithms as used in carrier recovery. Furthermore, this should be true even if IUD selection of symbols is being used.

Simulation Results

We have simulated a WDM system using the split-step Fourier method with adaptive step sizes[9]. We assume that five WDM channels travel along the fiber without any adds and drops. To properly simulate single-polarized data transmission over the optical fiber, we take into account the attenuation of the fiber. It is assumed that an erbium-doped fiber amplifier (EDFA) is located at the end of each span of length 50 km. Pulses are shaped as root-raised cosine pulses with a roll-off factor of about 6%. Channel spacing is 50 GHz, including about 6% guard band. Among the five WDM channels considered, the channel of interest, as before, is the middle one.

Two detection methods are considered. The first one consists of detection using a matched filter. In the second one, the channel of interest is fully back-propagated after being selected by a low-pass filter at the receiver. This is then followed by matched filtering and sampling. Phase compensation is done by a common phase rotation applied to all symbols chosen so that the average residual phase of the whole sequence of symbols is 0. An IUD transmission with a square 64-QAM, as well as a CC code of blocklength 171 (so that the transmission rates are comparable) are considered. The alphabet used for the CC code is selected by picking 171 points of a 256-QAM constellation having the least energy. The constellation figure of merit[10] of the CC code is 1.1 dB *worse* than the QAM constellation, i.e., use of the CC code entails a coding and shaping *loss* relative to QAM.

Under the same setup, a different decoder that incorporates a blind phase search (BPS) algorithm[11] at the very last step to minimize the phase error is considered. The decoder used is a genie-aided one as it uses the transmitted symbols to perform the best phase compensation that one might expect from the blind phase search algorithm. When detecting the symbol of interest $X_0 = a_{0,0}$, the BPS is done by solving the following minimization problem:

$$\phi_0 = \arg\min_{\theta} \sum_{j=-N}^{N} |X_j - Y_j e^{i\theta}|^2, \tag{4}$$

in which X_j is the j^{th} transmitted symbol and Y_j is the corresponding output of the matched filter. The output of the matched filter is rotated by ϕ_0 and the output of the BPS block is $Y_0 e^{i\phi_0}$. The window size of the BPS block is set to $2N+1 = 21$.

The SNRs obtained for the CC code of blocklength 171 and the IUD transmission from a 64-QAM constellation are shown in Fig. 2 when no carrier recovery is in place. The results of detection with BPS are shown in Fig. 3. The SNR gain of CC codes in the absence of BPS is about 0.3 dB without back-propagation and about 0.2 dB with back-propagation. With BPS, the SNR gain of CC codes is negligible (about 0.05 dB).

Conclusions

We have shown why short constant-composition codes reduce nonlinear interference noise in a WDM system. We have also shown that phase tracking can be used to achieve the same reduction in nonlinear interference noise even without using constant-composition codes.

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