# Coded Modulation for Four-Dimensional Signal Constellations with Concatenated Non-Binary Forward Error Correction


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***Abstract*** *A concatenated coded-modulation scheme for 4D constellations is proposed that consists of an inner complexity-optimized non-binary LDPC code and an outer zipper code. The related packing, shaping, and coding gains reduce the required SNR by 1 dB over conventional BICM with DP-16QAM.*


## Introduction

In dual-polarization (DP) optical systems, four-dimensional (4D) modulation schemes can provide an improved power efficiency[1],[2]. In particular, signal constellations based on the checkerboard lattice $\boldsymbol{D}_4$ enable a *packing gain*[3]–[5] over conventional DP quadrature-amplitude modulation (DP-QAM). Moreover, spherically-bounded 4D constellations, comprising points within a 4D hypersphere instead of a hypercube, readily provide an additional *shaping gain*[5],[6].

Recently, a coded-modulation (CM) scheme was proposed[4],[5] that makes effective use of $\boldsymbol{D}_4$-based constellations with soft-decision (SD) forward error correction (FEC). Still, the *two distinct* binary component codes used in this approach (i) are generic low-density parity-check (LDPC) codes that are not complexity-optimized, (ii) have to be successively decoded, and (iii) are not tailored to a CM approach that employs spherically-bounded (inherently-shaped) 4D constellations.

In this work, a concatenated CM architecture is proposed that is carefully designed to deliver the inherent packing and shaping gains of $\boldsymbol{D}_4$-based constellations. Following the philosophy of multi-level coding[7] (MLC), we use a 4D set partitioning (SP) to construct a few bit levels with low reliability, thereby maximizing the reliability of the other bit levels. The less reliable levels are then protected by a *single* inner, *non-binary* code, whereas the more reliable ones are only protected by the outer code. To this end, the binary code design from[8] is adapted to obtain a non-binary LDPC code with minimized decoding data flow. This inner code is tasked to reduce the bit error ratio (BER) on the bits passed to the hard-decision (HD) decoding for the outer zipper code[9] beneath its threshold, enabling the outer decoder to bring the total BER below $10^{-15}$.

The proposed CM scheme is assessed over the additive white Gaussian noise (AWGN) channel for an information rate of $R_\mathrm{i} = 6.971\ \mathrm{bit/symbol}$, compatible with the Implementation Agreement (IA) 400ZR[10]. When compared to conventional bit-interleaved coded modulation (BICM) for DP-16QAM, the 4D CM technique achieves a total gain in signal-to-noise ratio (SNR) of about $1\ \mathrm{dB}$, with only 4 (non-reliable) bit levels protected by an inner, complexity-optimized 16-ary LDPC code.

## Four-Dimensional Signal Constellations

We briefly review the construction and properties of 4D signal sets. In Fig. 1, their constellation-constrained capacities (in $\mathrm{bit/symbol}$) are plotted. Additionally, we show a two-dimensional projection of the related signal points, see also[5].

DP-16QAM is illustrated as the reference approach. When we normalize the constellation's squared minimum (4D) distance to $d^2_\mathsf{min} = 1$, its points form a subset of the algebraic structure of *Lipschitz integers*[3] (blue crosses in Fig. 1). We denote this particular type as *Lipschitz constellation*[4] $\mathcal{L}_M$ with 4D cardinality $M$ ($M = 16^2 = 256$ for DP-16QAM). If the cardinality is increased to $M = 1024$ (DP-32-cross-QAM) or $M = 4096$ (DP-64QAM), an SNR gain of about $0.7\ \mathrm{dB}$ is achieved for the target rate $R_\mathrm{i}$. However, since a *larger hypervolume* is covered in 4D space by the increased number of signal points, this SNR gain comes at the cost of an increased implementation penalty (e.g., due to the analog front end).

In 4D space, the number of points can be doubled within the *same hypervolume* without a decrease in minimum distance. The related algebraic structure with $d^2_\mathsf{min} = 1$, which is isomorphic to the $\boldsymbol{D}_4$ lattice, is known as *Hurwitz integers*[3]. A 512-ary *Hurwitz constellation*[4] (red dots in Fig. 1) can be constructed from a 256-ary

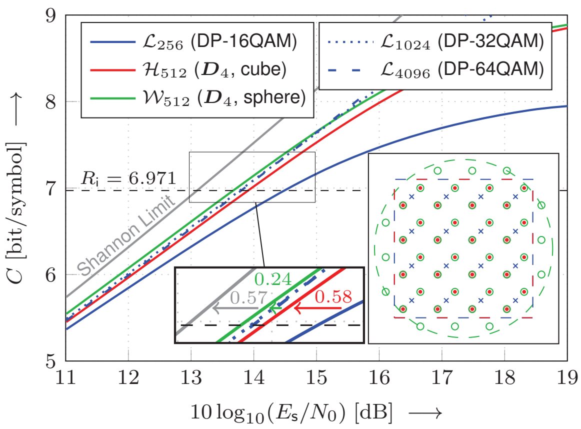


**Fig. 1:** Constellation-constrained capacities in bit/symbol versus the SNR in dB (energy per 4D symbol $E_\mathrm{s}$ over the one-sided noise power density $N_0$). The inset shows the 2D projection of the signal points of $\mathcal{L}_{256}$, $\mathcal{H}_{512}$, and $\mathcal{W}_{512}$.

Lipschitz constellation. This type of constellation is also known as 512-SP-QAM obtained by extension[2]. The additional bit enables a *packing gain* of $0.58$ dB at almost no additional implementation penalty compared to DP-16QAM[5].

The signal points of $\mathcal{L}_{256}$ and $\mathcal{H}_{512}$ are located within the same 4D *hypercube*, cf. Fig. 1. As proposed by G. Welti[6], the performance can further be improved by drawing subsets of the Hurwitz integers within a *4D hypersphere*. The resulting *Welti constellation* $\mathcal{W}_{512}$ (green circles) achieves a *shaping gain* of $0.24$ dB over $\mathcal{H}_{512}$. Without the need for any additional architectural complexity, even the capacity of DP-64QAM is surpassed and the Shannon limit is approached to within $0.57$ dB.

## Four-Dimensional Set Partitioning

The $\boldsymbol{D}_4$ lattice can be partitioned according to[11] $\boldsymbol{D}_4 \rightarrow \mathbb{Z}^4 \rightarrow \sqrt{2}\boldsymbol{D}_4 \rightarrow \sqrt{2}\mathbb{Z}^4 \rightarrow 2\boldsymbol{D}_4 \rightarrow \ldots$, i.e., into alternating, scaled (and rotated) variants of the 4D integer lattice $\mathbb{Z}^4$ and $\boldsymbol{D}_4$ itself. For $\mathcal{H}_M$ and $\mathcal{W}_M$ (isomorphic subsets of $\boldsymbol{D}_4$ with initial squared minimum 4D distance $d^2_{\mathrm{min},0} = 1$), a respective binary SP chain can be derived. The related chain for $\mathcal{W}_{512}$ is illustrated in Fig. 2: In the 1st partitioning step, $\mathcal{W}_{512}$ is decomposed into

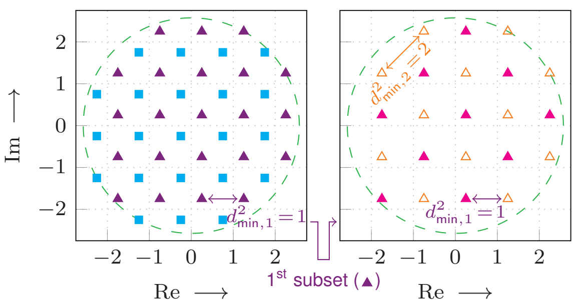


**Fig. 2:** Illustration of first (left) and second (right) partitioning step for the 512-ary Welti constellation $\mathcal{W}_{512}$. Real and imaginary part of the first polarization are shown; real and imaginary part in the second polarization are fixed to $\pm 1/4$.

two 256-ary $\mathbb{Z}^4$-based subsets (triangles versus squares), still with $d^2_{\mathrm{min},1} = 1$. In the 2nd step, 128-ary $\boldsymbol{D}_4$-based subsets with the squared minimum intra-set distance $d^2_{\mathrm{min},2} = (\sqrt{2})^2 = 2$ are obtained. Following this partitioning chain, in every 2nd level, $d^2_{\mathrm{min}}$ grows by a factor of $2$. As shown below, the value $d^2_{\mathrm{min},4} = 4$ after 4 levels can already enable highly reliable uncoded transmission.

## Concatenated Non-Binary FEC Architecture

The proposed concatenated FEC approach is shown in Fig. 3. The generic model is illustrated for a 512-ary constellation $\mathcal{A}$, i.e., $\mathcal{H}_{512}$ or $\mathcal{W}_{512}$.

First, the message sequence $\langle \mathrm{q} \rangle$ with elements drawn from the binary field $\mathbb{F}_2$ is encoded with the outer code (ENC$^\mathrm{out}$). The resulting stream is parallelized into $\log_2(M) = 9$ bit levels $\tilde{\mathrm{q}}_0, \ldots, \tilde{\mathrm{q}}_8$. Then, the $b_\mathrm{c} = 4$ least reliable (lowest) levels are additionally encoded with the inner code (ENC$^\mathrm{in}$). In contrast to conventional MLC[7], where $b_\mathrm{c}$ binary encoders are used, a *single non-binary* code over the extension field $\mathbb{F}_{2^{b_\mathrm{c}}}$ is applied. The $b_\mathrm{u} = 5$ upper (reliable) levels are directly passed to the constellation mapper $\mathcal{M}$. Thereby, a *mixed partitioning*[7] is used, where the first $b_\mathrm{c} = 4$ bit levels are labeled according to the above SP chain, whereas a *pseudo-Gray labeling*[12] is employed in the upper (reliable) $b_\mathrm{u} = 5$ levels. The $b_\mathrm{c}+b_\mathrm{u} = 9$ bits are jointly assigned to a 4D symbol $\boldsymbol{a} \in \mathcal{A}$.

DP transmission over the AWGN channel models the linear regime of the optical fiber. Noisy 4D symbols $\boldsymbol{y} = \boldsymbol{a} + \boldsymbol{n}$ are obtained, where $\boldsymbol{n}$ represents 4D noise with independent components.

At the receiver, non-binary SD decoding is performed to reconstruct the $b_\mathrm{c}$ levels protected by the inner code (DEC$^\mathrm{in}$). The decoded bits $\hat{\tilde{\mathrm{q}}}_0, \ldots, \hat{\tilde{\mathrm{q}}}_3$ address one of the $2^{b_\mathrm{c}} = 16$ constellation subsets after 4 SP steps; their $2^{b_\mathrm{u}} = 32$ points represent the 5 uncoded bits. They are jointly estimated ($\hat{\tilde{\mathrm{q}}}_4, \ldots, \hat{\tilde{\mathrm{q}}}_8$) by a simple quantization of $\boldsymbol{y}$ to the nearest point within the subset (QUAN); they are hence available without any decoding delay. After parallel-to-serial conversion, the outer HD decoder (DEC$^\mathrm{out}$) handles the residual bit errors and provides the estimated bit sequence $\langle \hat{\mathrm{q}} \rangle$.

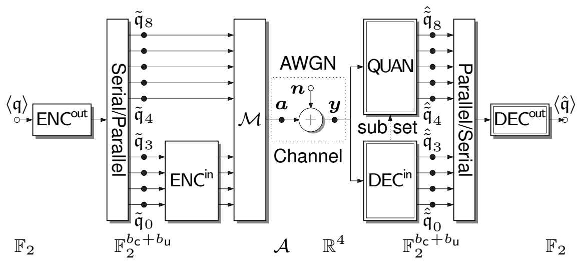


**Fig. 3:** System model of the concatenated non-binary FEC architecture for DP transmission over the AWGN channel.

**Tab. 1:** Bit-level capacities at the inner information rate of $R_\mathrm{i}^\mathrm{in} = 7.187$ bit/symbol and related inner code rate $R_\mathrm{c}^\mathrm{in}$.

| Level | Labeling | $d^2_{\mathrm{min}}$ | $\mathcal{L}_{256}$ | $\mathcal{H}_{512}$ | $\mathcal{W}_{512}$ |
|---|---|---|---|---|---|
| 8 | Pseudo Gray | 4 | 0.999 | 0.998 | 0.998 |
| 7 | | | 0.999 | 0.997 | 0.997 |
| 6 | | | 0.999 | 0.995 | 0.996 |
| 5 | | | 0.998 | 0.994 | 0.994 |
| 4 | | | 0.998 | 0.992 | 0.992 |
| 3 | Set Partitioning | 2 | 0.904 | 0.817 | 0.830 |
| 2 | | | 0.857 | 0.746 | 0.762 |
| 1 | | 1 | 0.433 | 0.279 | 0.359 |
| 0 | | | – | 0.369 | 0.258 |
| $R_\mathrm{c}^\mathrm{in}$ (level 4–8 uncoded) | | | 0.729 | 0.547 | |

In contrast to BICM, the inner, non-binary CM approach is optimal from an information-theoretic point of view and is thus suited to exploit the 4D constellations' capacities. This benefit comes at the expense of high decoding complexity[13],[14]. To address that issue, the design approach from[8] originally defined for binary codes is adapted to obtain non-binary LDPC codes with minimal decoding data flow. Besides, by only encoding the lowest $b_\mathrm{c} = 4$ levels, the field size ($\mathbb{F}_{16}$) is kept small. In this work, we restrict to conventional (non-approximate) message passing[13],[14].

The 16-ary LDPC code (of rate $R_\mathrm{c}^\mathrm{in} = 0.547$) enables—along with the 5 uncoded levels—the inner information rate of $R_\mathrm{i}^\mathrm{in} = 4 \cdot 0.547 + 5 = 7.187$ bit/symbol. The respective *bit-level capacities*[7] of the considered constellations are listed in Tab. 1. As expected, the 4D SP leads to large capacity gains after the levels $1$ and $3$. The capacities of the pseudo-Gray-labeled levels are close to one; they can be left uncoded. The inner CM scheme is concatenated to an outer binary zipper code[9] (rate $R_\mathrm{c}^\mathrm{out} = 0.97$), altogether leading to $R_\mathrm{i} = 7.187 \cdot 0.97 = 6.971$ bit/symbol as defined in the IA 400ZR[10]. A similar architecture is possible for $\mathcal{L}_{256}$ (DP-16QAM), where—due to the $\mathbb{Z}^4/\boldsymbol{D}_4$ SP chain—3 coded levels are sufficient.[1]

## Numerical Results

Simulations over the AWGN channel with an inner frame length of 6000 symbols—equal to the length of the non-binary LDPC code—have been conducted. The proposed architecture is assessed for $\mathcal{L}_{256}$, $\mathcal{H}_{512}$, and $\mathcal{W}_{512}$ and compared to the following binary schemes: (i) concatenated FEC based on 1D SP for $\mathcal{L}_{256}$ with an optimized binary inner LDPC code[8], (ii) BICM for $\mathcal{L}_{256}$ with a generic binary LDPC code[5], and (iii) two-stage BICM (TS-BICM) for $\mathcal{H}_{512}$ with two binary generic LDPC codes[5]. The binary code lengths were chosen such that the same frame length was covered. In general, floating-point sum-product message passing with 10 iterations was performed.

[1] In the IA 400ZR, DP-16QAM is deployed with the code rates $R_\mathrm{c}^\mathrm{in} = 119/128$ and $R_\mathrm{c}^\mathrm{out} = 239/255$. As a more sophisticated outer code is used in this work, the inner information rate $R_\mathrm{i}^\mathrm{in}$ is slightly modified to enable the same total rate $R_\mathrm{i}$.

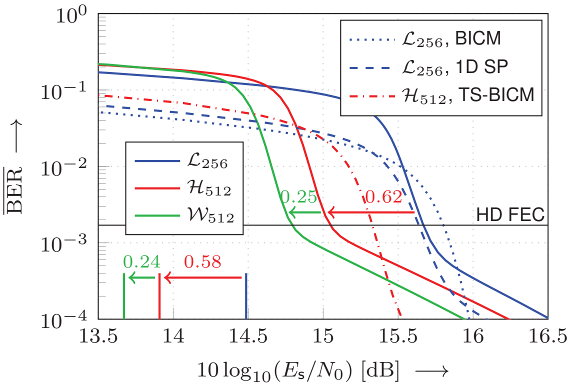


**Fig. 4:** Average BER on the bits passed to the outer decoder versus the SNR in dB ($10^5$ frames). Concatenated non-binary FEC according to Fig. 3 and Tab. 1 (solid lines) and various binary approaches (dotted and/or dashed lines) are applied. The capacities from Fig. 1 are indicated by vertical lines. The horizontal line represents the threshold for the outer code.

In Fig. 4, the average BER on the bits passed to the outer decoder is plotted; this includes the bits obtained from inner decoding as well as the uncoded ones, cf. Fig. 3. A conservative error rate of $\overline{\mathrm{BER}} < 1.7 \cdot 10^{-3}$ is targeted; below that threshold, the outer zipper code can bring the BER below $10^{-15}$, cf.[9]. At the target BER, with $\mathcal{L}_{256}$, the proposed non-binary scheme (3 coded levels) performs almost as well as the binary 1D SP approach (4 coded levels). Both approaches exhibit a gain of about $0.15$ dB over BICM (8 coded levels). With $\mathcal{H}_{512}$, the non-binary scheme enables $0.62$ dB SNR gain, even a little bit more than the expected packing gain of $0.58$ dB. The related TS-BICM scheme[5] (joint BICM in 2nd stage) shows a significant loss, mainly caused by error propagation from the 1st stage in the scenario at hand. With $\mathcal{W}_{512}$, the non-binary approach achieves the expected shaping gain of about $0.25$ dB over $\mathcal{H}_{512}$. A total *shaping, packing, and coding gain* of around $1$ dB over conventional BICM is present.

## Conclusion and Outlook

A concatenated FEC scheme with complexity-optimized non-binary inner LDPC code has been proposed. In combination with 4D constellations, gains of up to $1$ dB over conventional BICM and DP-16QAM are enabled. A detailed complexity study including suboptimal low-complexity decoding strategies[14]–[16] will be a topic of future work.


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