A Novel Approach to Coupled-Mode Analysis of Geometric Deformations in Reciprocal Waveguides

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Abstract *We present an approach to treat geometric perturbations in coupled-mode analysis for reciprocal waveguides. Geometric perturbations are converted into material ones and a coupled-mode theory for material perturbations is used to assess the propagation. The method is tested on an elliptically deformed antiresonant fiber.*

Introduction

Optical waveguides are designed to guide light with precise properties. However, imperfections introduced from manufacturing or their deployment conditions induce perturbations that may impair their behavior. Assessing how these perturbations affect the propagation is thus of paramount importance for real applications.

In this context, *coupled-mode theory* (CMT)[1] is a valuable tool to determine the optical properties of real waveguides. It describes light propagation by using the modes of an unperturbed, *reference waveguide*, which are readily available from the design phase. Since these modes are not exact solutions in the real waveguide, their amplitudes are coupled in the propagation.

So far, many CMTs have been presented in the literature, encompassing a variety of approaches and formulas.[2]-[4] Choosing the right one is not an easy task and depends on the waveguide, the nature of the modes, and the perturbations.

In general, perturbations can be classified as *material* or *geometric*. *Material* perturbations change the constitutive tensors of the waveguide without affecting the geometry, whereas, geometric perturbations deform the waveguide's structure. This distinction splits CMTs into two *separate* categories, corresponding to the type of the perturbation they aim to tackle. The reason involves the distinctive behavior of geometric deformations, which shift the boundary of adjacent materials, misaligning the wave in the waveguide with the modes of the reference that are used for its approximation. This critical problem requires dedicated CMT formulations to handle geometric deformations accurately, especially for high-index contrast waveguides.[5],[6]

CMTs for material perturbations and those for geometric ones, in general, are not interchangeable, so the appropriate CMT must be chosen depending on the perturbation type. Unfortunately, real perturbations are neither only electromagnetic nor only geometric but a mixture of both. Therefore, these case can only be analyzed by resorting to finite-element method (FEM) analysis. Although FEM is a great tool for designing waveguides, it is not well suited to assess the effects of perturbations since it requires that the entire wave be computed anew for each waveguide variation.

Here, we present a novel approach to the coupled-mode analysis of reciprocal waveguides affected by geometric and material perturbations. Geometric perturbations are first converted into material ones by using the theory of transformation optics (TO). [7],[8] Then, a newly-developed, general CMT for material perturbations is applied to study propagation. This approach has the advantage of putting geometric and material perturbations on the same footing, enabling a unified coupled-mode theory for treating both geometric and material perturbations simultaneously. Moreover, our method can be used for any kind of reciprocal waveguides, for both guided and leaky modes, including micro-structured,[9] solid-core and antiresonant fibers,[10] as well as high-contrast integrated waveguides.[11]

Results

We consider a *waveguide* affected by material and geometric perturbations, with permittivity and permeability tensors $\hat{\boldsymbol{\varepsilon}}(\boldsymbol{x})$ and $\hat{\boldsymbol{\mu}}(\boldsymbol{x})$, where $\boldsymbol{x} = \{x_1, x_2, x_3\}$ is the coordinate vector in a Cartesian reference frame $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$. To study the waveguide's optical properties, we first define an *equivalent waveguide* in which the geometric perturbation is converted into a material one. This is obtained by applying a proper change of coordinates defined by $\boldsymbol{q} = \boldsymbol{\sigma}(\boldsymbol{x})$. Accordingly, the transformed space has coordinate vector $\boldsymbol{q} = \{q_1, q_2, q_3\}$ in the Cartesian reference frame

$\{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3\}$. This transformation is carefully selected so that the equivalent waveguide and the reference one share the same boundaries. Owing to TO, for the electromagnetic wave before and after the transformation to be *invariant*, the equivalent waveguide must have constitutive tensors, $\boldsymbol{\epsilon}(\boldsymbol{q})$ and $\boldsymbol{\mu}(\boldsymbol{q})$, given by[8]

$$\boldsymbol{\epsilon} = (\det \boldsymbol{J}_\sigma)^{-1}\, \boldsymbol{J}_\sigma\, \hat{\boldsymbol{\epsilon}}\, \boldsymbol{J}_\sigma^{\mathrm{T}}, \tag{1}$$

$$\boldsymbol{\mu} = (\det \boldsymbol{J}_\sigma)^{-1}\, \boldsymbol{J}_\sigma\, \hat{\boldsymbol{\mu}}\, \boldsymbol{J}_\sigma^{\mathrm{T}}, \tag{2}$$

where $\boldsymbol{J}_\sigma$ is the Jacobian matrix of $\boldsymbol{\sigma}$. In fact, TO converts geometric deformations into anisotropic changes of the material. Remarkably, note that $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ may also include any material perturbation possibly present in the original waveguide. Moreover, if the waveguide is not affected by geometric deformations, i.e., $\boldsymbol{\sigma}$ is the identity function, then $\boldsymbol{\epsilon} = \hat{\boldsymbol{\epsilon}}$ and $\boldsymbol{\mu} = \hat{\boldsymbol{\mu}}$; thus, the equivalent waveguide retains the initial perturbation on the material.

Once the equivalent waveguide is defined, CMT is used to assess its behavior. Nevertheless, classic CMTs consider only perturbations on the permittivity of the material; thus, they cannot be used in this case because both the dielectric permittivity and magnetic permeability tensors are perturbed. Here, we present a general CMT for reciprocal waveguides able to deal with perturbations on both tensors. As the derivation of this theory is lengthy, details will be reported elsewhere; in this work, we report the main results.

The electromagnetic wave in the equivalent waveguide is described by a combination of the modes of the reference one, according to the *coupled-mode equation* of propagation. This differential equation reads

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}z} = -j\,\{\, \boldsymbol{D} + \boldsymbol{Q}^{-1}[\boldsymbol{K}(z) + \boldsymbol{C}(z)]\,\}\,\boldsymbol{a}, \tag{3}$$

where $\boldsymbol{a}(z)$ are the amplitudes of the modes; $\boldsymbol{D}$ is the diagonal matrix of the modes' propagation constants; $\boldsymbol{Q}$ is the orthogonality matrix, whose elements are the mode orthogonality coefficients; and $\boldsymbol{K}(z)$ and $\boldsymbol{C}(z)$ are the coupling matrices, whose elements are the mode coupling coefficients. $\boldsymbol{K}(z)$ and $\boldsymbol{C}(z)$ represent the local mode coupling due to perturbation on the permittivity and permeability tensor, respectively. Here, both contributions are crucial to describe the correct behavior of the waveguide.

The *coupling coefficients* in $\boldsymbol{K}$ are denoted as $K_{\xi\ \nu}^{(q)(p)}(z)$ and describe the interactions that occur between the generic modes $\xi^{(q)}$ and $\nu^{(p)}$, where the superscripts, (q) and (p), indicate the mode's direction of propagation—both forward and backward propagating modes are considered in the analysis. The expression of the coefficients is

$$K_{\xi\ \nu}^{(q)(p)} = \omega \iint \left(\boldsymbol{T}\boldsymbol{e}_\xi^{(q)}\right)^{\mathrm{T}} \boldsymbol{M}(z) \left(\boldsymbol{T}\boldsymbol{e}_\nu^{(p)}\right) dA, \tag{4}$$

where the integral is over the waveguide's cross-section; ω is the angular frequency; $\boldsymbol{e}$ is the electric field of the indicated mode; $\boldsymbol{T} = \mathbf{I} + \bar{\boldsymbol{\epsilon}}_{zt}/\bar{\epsilon}_{zz}$, with $\mathbf{I}$ the identity matrix, is a 3×3 matrix that depends on the permittivity tensor $\bar{\boldsymbol{\epsilon}}$ of the reference waveguide; and $\boldsymbol{M}(z)$ is a 3×3 matrix that depends on the difference between the equivalent waveguide and the reference one,

$$\begin{aligned} \boldsymbol{M}(z) = (\boldsymbol{\epsilon}_{tt} - \bar{\boldsymbol{\epsilon}}_{tt}) - \left(\frac{\boldsymbol{\epsilon}_{tz}\cdot\boldsymbol{\epsilon}_{zt}}{\epsilon_{zz}} - \frac{\bar{\boldsymbol{\epsilon}}_{tz}\cdot\bar{\boldsymbol{\epsilon}}_{zt}}{\bar{\epsilon}_{zz}}\right) \\ + \left(\frac{\bar{\epsilon}_{zz}}{\epsilon_{zz}}\boldsymbol{\epsilon}_{tz} - \bar{\boldsymbol{\epsilon}}_{tz}\right) + \left(\frac{\bar{\epsilon}_{zz}}{\epsilon_{zz}}\boldsymbol{\epsilon}_{zt} - \bar{\boldsymbol{\epsilon}}_{zt}\right) \\ + \frac{\bar{\epsilon}_{zz}}{\epsilon_{zz}}(\boldsymbol{\epsilon}_{zz} - \bar{\boldsymbol{\epsilon}}_{zz}). \end{aligned} \tag{5}$$

In the above expressions, $\boldsymbol{\epsilon}_{tt}$, $\boldsymbol{\epsilon}_{tz}$, $\boldsymbol{\epsilon}_{zt}$, $\boldsymbol{\epsilon}_{zz}$, and the equivalent ones for the reference waveguides are 3×3 matrices where the only non-zero elements are indicated in the subscripts;[3] ϵ_{zz} and $\bar{\epsilon}_{zz}$ are scalars that indicate the element in position (3,3) of $\boldsymbol{\epsilon}$ and $\bar{\boldsymbol{\epsilon}}$, respectively. Regarding the coupling coefficients of $\boldsymbol{C}$, analogous expressions hold once $\boldsymbol{e}$ is replaced with the magnetic field $\boldsymbol{h}$, $\boldsymbol{\epsilon}$ is replaced with $\boldsymbol{\mu}$, $\bar{\boldsymbol{\epsilon}}$ with the reference waveguide's permeability tensor $\bar{\boldsymbol{\mu}}$, and $\boldsymbol{M}(z)$ with $-\boldsymbol{M}(z)$.

The *orthogonality coefficients* in $\boldsymbol{Q}$ read

$$Q_{\xi\ \nu}^{(q)(p)} = \iint \hat{\boldsymbol{z}} \cdot \left(\boldsymbol{e}_\xi^{(q)} \times \boldsymbol{h}_\nu^{(p)} - \boldsymbol{e}_\nu^{(p)} \times \boldsymbol{h}_\xi^{(q)}\right) dA, \tag{6}$$

where $\hat{\boldsymbol{z}}$ is the unit vector in the longitudinal direction. In reciprocal waveguides, these coefficients are non-zero only if $\xi = \nu$ and $(q) = (\bar{p})$, where $(\bar{p})$ indicates the counter-propagating direction of (p). Remarkably, this *orthogonal relation* holds for any kind of mode, including guided, leaky, and radiation ones.

Case study

The presented method is used to analyze the propagation in a circular 2-layer antiresonant hollow-core fiber[10] (i.e., a capillary fiber made of concentric rings) affected by ellipticity. In this preliminary application, we keep the ellipticity fixed along the longitudinal direction.

The unperturbed, *reference fiber* is isotropic, with $\bar{\mu} = \mu_0$ and dielectric profile $\bar{\epsilon}$ (see Tab. 1). Being hollow-core, the propagation is described by leaky modes.[12] Thus, a perfectly matched layer (PML) is used to compute the modes of the fiber with a FEM solver (i.e., COMSOL®).

The deformation which shifts the fiber's structure from circular to elliptical is described by

Tab. 1: Concentric rings of the 2-layer antiresonant fiber

	Radii $[\mu m]$ left	Radii $[\mu m]$ right	Refractive Index
Hollow-core	0	23.2	1
1st antiresonant layer	23.2	24.3	1.444
2nd antiresonant layer	24.3	39.5	1
Cladding	39.5	70	1.444
PML	70	90	1.444

the parameter γ, defined as the relative maximum radius variation with respect to the circular shape. The *elliptical fiber* is considered with its major axis aligned with $\boldsymbol{x}_1$, in the Cartesian reference frame $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$, where $\boldsymbol{x}_3$ points to the longitudinal direction. Accordingly, the coordinates change $\boldsymbol{\sigma}$ that recovers the fiber's roundness is $q_1 = x_1/(1+\gamma)$, $q_2 = x_2/(1-\gamma)$, and $q_3 = x_3$. Resorting to Eqs. (1) and (2), the constitutive tensors of the *equivalent fiber* are $\boldsymbol{\epsilon} = \boldsymbol{\Lambda}\,\bar{\epsilon}$ and $\boldsymbol{\mu} = \boldsymbol{\Lambda}\,\mu_0$, where $\boldsymbol{\Lambda} = (\det \boldsymbol{J}_{\sigma})^{-1}\,\boldsymbol{J}_{\sigma}\boldsymbol{J}_{\sigma}^{\mathrm{T}} = \mathrm{diag}(\Lambda_1, \Lambda_2, \Lambda_3)$, with $\Lambda_1 = (1-\gamma)/(1+\gamma)$, $\Lambda_2 = (1+\gamma)/(1-\gamma)$, and $\Lambda_3 = 1-\gamma^2$.

Once the equivalent fiber has been defined, the propagation in the elliptical one is described by the coupled-mode equation of Eq. (3). Since ellipticity is fixed along the fiber, the coupling matrices $\boldsymbol{K}$ and $\boldsymbol{C}$ are constants. Then, the solutions of the coupled-mode equation are easy to find and they correspond to the eigenmodes of the elliptical waveguide.

To evaluate the soundness of our approach, the obtained eigenmodes are compared with those computed with COMSOL® from the original elliptical waveguide, which are used as a benchmark. Furthermore, we also compare our method with the standard CMT for leaky modes,[13] with the perturbation on the dielectric profile given by the difference between the profile of the elliptical fiber and the one of the reference. For both, the results are obtained by using the 100 lowest loss modes of the reference fiber.

Fig.1 shows the real part of the effective refractive index and the loss of the six lowest loss eigenmodes of the elliptical fiber at different values of γ. Solid blue lines represent the results obtained with the method presented in this paper; dashed red lines indicate those obtained with the standard CMT; and the black cross marks are the benchmark values. Results show that our approach performs consistently better than the standard CMT, which completely fails for high ellipticity values. The reason is that standard CMT does not account for the boundary shifts induced by the ellipticity. The issue is likely to be aggravated by the fact that leaky modes are in general not well confined in the core. When compared with the benchmark values, our results are in very good agreement, especially for the real part of the effective refractive index. Concerning the losses, results follow the benchmark until $\gamma = 10^{-2}$, beyond which they worsen. We believe, however, that this is not a limitation of the theory, but rather an effect of the limited number of modes used in the analysis.

Conclusions

In this work we have presented a novel approach based on the coupled-mode analysis to study propagation in reciprocal waveguides affected by geometric and material perturbations. The method paves the way for a unified CMT able to treat geometric and material perturbation in the same framework. To prove its soundness, we tested our approach on a 2-layer antiresonant fiber affected by ellipticity. Results show that the method performs better than the standard CMT and very close to the reference values.

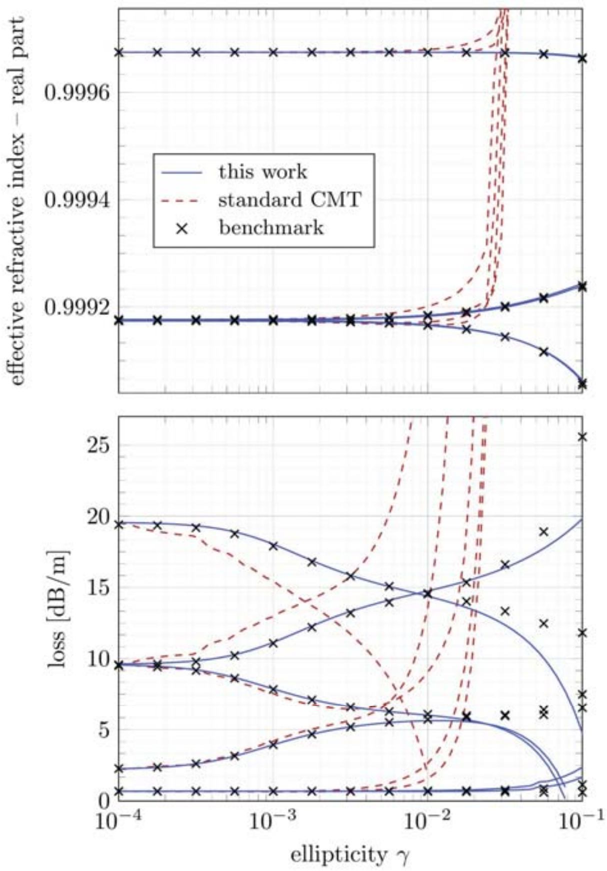

Fig. 1: Effective refractive index and losses of the six lowest loss eigenmodes of the elliptical fiber.

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